

XV. *A new Theory of the Rotatory Motion of Bodies affected by Forces disturbing such Motion.* By Mr. John Landen, F. R. S.

Read Feb. 20, 1777. **I** AM induced to consider this paper as not unworthy the notice of this Society, through a persuasion that the theory herein contained will conduce to the improvement of science, by enabling the reader to form a true idea, and accordingly to make a computation of the motion (or change) of the axis about which a body having a rotatory motion will turn, or have a tendency to turn, upon being affected by a force disturbing its rotation; particularly of the motion of the earth's axis arising from the attraction of the Sun and Moon on the protuberant matter of the earth above its greatest inscribed sphere: which compound motion, I conceive, has not been rightly explained by any one of the eminent mathematicians whose writings on the same subject have come to my hands. Whether in this essay I have really succeeded better than other writers who have attempted an explanation of such motion, I submit to gentlemen well versed in mechanics to determine.

I. Fig.

1. Fig. 1. Let the sphere $ADBE$, whose radius is r , revolve uniformly about the diameter ACB as an axis, with the angular velocity c , measured at D or E , the motion being according to the order of the letters $DGEH$ in the section at right angles to ACB , fig. 2.; and, whilst it is so revolving, let the pole A be impelled by some instantaneous percussive force to turn about the diameter DCE , from A towards H , with the velocity w . It is proposed to find the new axis about which the sphere will revolve after receiving such impulse.

Calling al , parallel to DC , x ; cl will be $=\sqrt{r^2-x^2}$: the velocity of the point z (about ACB) before the impulse on A will be $=\frac{cx}{r}$; and the velocity (about DCE) given to the same point (a) by the said impulse will be $=\frac{w\sqrt{r^2-x^2}}{r}$. Which velocities of the point a being in contrary directions, if it be so situated that they be equal, then, one destroying the other, that point will stop and become one of the new poles sought, about which the former poles A and B will revolve with the velocity w ; and the points D and E will revolve with the same velocity (c) as before the perturbing impulse on the point A ; but instead of describing the great circle $DGEH$, their motion will be about the new axis ab ; about which they (as well as the points A and B) will describe lesser circles parallel

to the great circle de , in which the points d and e (de being at right angles to ab) will revolve about the same axis (ab) with the velocity $\sqrt{c^2 + w^2}$. Which being denoted by e , and m and n being put for the sine and cosine of the angle $Ac a$ to the radius r , me will be $=w$, $ne=c$, and consequently $mn e^2 = cw$.

Now taking $\frac{cx}{r} = \frac{w\sqrt{r^2 - x^2}}{r}$, in order to find that new axis ab , we have from that equation $x = \frac{rw}{\sqrt{c^2 + w^2}} = \frac{rw}{c} = al$.

Moreover it is obvious, that if a spheroid, a cylinder, or any other body, whose center of gravity is c and proper axis ACB , were, whilst revolving about that axis with the same angular velocity (c), to receive such an impulse as instantly to give the point A the angular velocity w about DCE ; the axis about which that spheroid, cylinder, or other body, immediately after the impulse, would revolve, or would have a tendency to revolve, would be the same line ab .

The great circle de (fig. 1.) and any other great circle so situated with respect to the axis of any revolving sphere, I shall denominate the *mid-circle*.

2. In the manner above described the poles of the sphere are by the instantaneous impulse on the point A instantly changed from A and B to a and b . But if, instead of such impulse, a continued attractive force F (like that of gravity)

vity) acted at A fig. 3. and at the new poles a' , a'' , &c. as they become such by a successive change caused by such continued action of the force F urging the sphere at every instant to revolve about the diameter de' , or de'' , &c. of the contemporary mid-circle, the new pole (a' , a'' , &c.) would not instantly be at a finite distance from the primitive pole A, but some finite time would be requisite, that by such successive change, the pole might be varied to a finite distance from A: and the force F continuing invariable, the velocity (v) wherewith the pole changed its place would be expressed by $\frac{x}{t}$, t being the time elapsed whilst the pole is varying from A to a , and x the length of the arc A a . Therefore the velocity wherewith the pole will change its place during such action of the force F will be expressed in the same manner as the velocity (v) of a body moving uniformly from A to a in the time t may be expressed; that is, in both cases v will be $= \frac{x}{t}$. But there is a material difference between the motion of a body so moving from A to a and the change of place of the pole a' , a'' , &c. the former is permanent, and will continue to carry the body forward without the action of any force whatever; whereas the latter will instantly cease, and the axis will keep its position, if the
force

force F ceases to act thereon; like as the varying direction of a projectile near the earth's surface would immediately cease to change, if the force of gravity ceased to act.

It is observable, that whilst the force F acts, and the revolving sphere, in consequence of such action, every moment takes a new axis, the angular motion about the axis will continue invariable; the action of such force only altering the axis without altering the angular velocity of the sphere about it: like as the direction of a moving body is altered, without altering the velocity thereof, by an attractive force continually acting on it in a direction at right angles to that in which the body is moving. And if ever the force F shall cease to act, the sphere will instantly revolve with its primitive velocity (c) about the axis it then may have been brought to take by the preaction of that force.

The new axis, about which the sphere has such tendency to revolve at any instant during the action of the force F , I shall call the *momentary axis*; and the poles thereof the *momentary poles*.

3. From the equation $\frac{cx}{r} = \frac{w\sqrt{r^2 - x^2}}{r}$ (art. 1.) we have

$\frac{w}{x} = \frac{c}{\sqrt{r^2 - x^2}}$. Now if a continued attractive force (F) act during the time t as above-mentioned, instead of the

instantaneous percussive force at A, we, according to the doctrine of fluxions, must, instead of w , take $\dot{w}$, or its equal $r \dot{z}$, and $\dot{x}$ instead of x , in the expression $\frac{w}{x}$; therefore, in this case, we have $\frac{\dot{w}}{\dot{x}} = \frac{r \dot{z}}{\dot{x}} = \frac{c}{\sqrt{r^2 - x^2}}$. Whence, putting z for the arc (A a' , or A a'' , &c.) whose sine is x , and writing $\dot{z}$ for its equal $\frac{r \dot{x}}{\sqrt{r^2 - x^2}}$, we get $\frac{r \dot{z}}{\dot{z}} = c$, or $\dot{z} = \frac{r \dot{z}}{c}$.

Hence v denoting the velocity wherewith the momentary pole (a' , a'' , &c.) changes its place during the action of the accelerative force F , we have $\dot{z} = v \dot{t} = \frac{r F \dot{t}}{c}$, and consequently $v = \frac{r F}{c}$.

4. The value of v may also be determined in the following manner (fig. 4.). Conceive a very thin string (without weight) to have one of its ends fastened to a fixed point l and the other to a heavy particle of matter m ; also conceive such particle so to revolve with the velocity e , about the line lm , that a certain accelerative force F (like that of gravity referred to a certain direction) continually acting on the said particle m , in a direction at right angles both to the string lm and to the tangent to the curve in which m is moving, the string shall describe a conical surface. Then lm being denoted by

by r , and mo , perpendicular to ln , by q ; $\frac{e^2}{q}$, the centrifugal force urging m in the direction om , will be to F as r to $\sqrt{r^2 - q^2} = lo$. Therefore F must be $= \frac{e^2 \sqrt{r^2 - q^2}}{rq}$. Now if, whilst m is so revolving, the force F ceases acting, the said particle (m) will, 'it is obvious, immediately proceed to describe a great circle of the sphere whose radius is r and center l , of which great circle one of the poles will be situated in a lesser circle parallel to, and 90° distant from, that described by m during such action of the said force; which pole, during such action, will change its place in the said lesser circle in which it will at any time be found with a velocity (v) which will be to e as (s) the radius of the last mentioned circle to q . But s will be $= \sqrt{r^2 - q^2}$; therefore we have $u : e :: \sqrt{r^2 - q^2} : q$, and $\frac{\sqrt{r^2 - q^2}}{q} = \frac{v}{e}$. Consequently $F = \frac{e^2 \sqrt{r^2 - q^2}}{rq}$ will be $= \frac{e^2}{r} \times \frac{v}{e} = \frac{ev}{r}$, and $v = \frac{rF}{e}$.

Let now m be a point on the surface of a sphere whose center is l , and radius $lm = r$; and let the sphere revolve about an axis so that m shall describe a great circle with the velocity e . If then such a motive force begins to act on the sphere, that, continuing its action, the point m shall always be urged by the invariable accelerative force F to move in a direction at right angles to the ray lm and

to the tangent to the curve which m will describe; that point it is obvious will, in consequence of the action of that force, describe a lesser circle of the same radius (q) as that described by the particle m when fastened to a string and acted on by the force F as above-mentioned; and the center of the sphere being always considered as at rest, one of the momentary poles of the sphere will describe a circle whose radius will be $= \sqrt{r^2 - q^2}$ parallel to, and 90° distant from, that described by the point m . For if the said force were to cease acting, that point of the sphere would describe a great circle, as would the particle m at the string in the like case; and therefore both the said particle and the point m of the sphere at every instant having the same tendency, and being acted on by equal accelerative forces, the effect will be the same with respect to the motion of each. Consequently, v being put to denote the velocity wherewith the momentary pole changes its place in the circle which it will describe whilst the motive force producing the accelerative force F acts on m as just now mentioned, v will be $= \frac{rF}{e}$, the same as in the preceding article, e here denoting that velocity which we there denoted by c .

5. Referring the point of action of the perturbing force to the mid-circle we have not hitherto considered

that point as varied with a greater or less velocity than (e) that of the point m ; that is, with reference to such circle we have always considered the point m as the point of action. But it is obvious, that, *ceteris paribus*, the point of action with respect to the mid-circle (which point we will now denote by q) may be varied with a velocity greater or less than e ; and that, *ceteris paribus*, the velocity (v) of the momentary pole will be the same with what velocity soever (q) the point of action of the force r be varied; the direction in which that force acts being always at right angles to the ray (lq) from the center of the sphere, and to the tangent to the curve described by (q) such point of action.

Yet, although v continues the same whether, *ceteris paribus*, (u) the velocity of the point q be greater, equal to, or less than e , the immoveable circle in which the momentary pole will be found will not continue the same; that circle being greater, equal to, or less than the circle whose radius is $\sqrt{r^2 - q^2}$ according as u is less, equal to, or greater than e , as will be made more evident by what follows.

6. Fig. 5. Let $\acute{p}$ (in the great circle $R\acute{p}Q\acute{q}T$) be one of the poles of the axis about which the sphere $RSTV$, whose radius is r , is revolving (according to the order of the letters $v\acute{q}s$) with the angular velocity e , measured at

the distance r from the axis; and whilst it is so revolving let the said pole be urged to turn about a diameter of the mid-circle vqs towards q , by an accelerative force F , and let such force continue to act on the successive new poles $\overset{''}{p}, \overset{'''}{p}, \&c.$ as they become such, always urging the sphere to turn about a diameter of the contemporary mid-circle, whilst the direction in which such perturbing force acts is regulated in the following manner.

Conceive the said revolving sphere to be surrounded by an immoveable concave sphere of the same radius r . Then the momentary pole ($\overset{'}{p}, \overset{''}{p}, \overset{'''}{p}, \&c.$) will always be found in some curve $\overset{'''''}{p} \overset{'''''}{p} \overset{'''''}{p} \&c.$ in the said concave sphere, and in some curve $\overset{'''''}{p} \overset{'''''}{p} \overset{'''''}{p} \&c.$ on the revolving sphere; which last mentioned curve will continually touch and roll along the other curve $\overset{'''''}{p} \overset{'''''}{p} \overset{'''''}{p} \&c.$ on the immoveable sphere, the force F and the direction in which it acts varying in any manner whatever. Let F be invariable; then, it is obvious, the two curves so touching each other will be circles; and if great circles $\overset{''}{p} \overset{''}{q}, \overset{''}{p} \overset{''}{q}, \overset{'''}{p} \overset{'''}{q}, \&c.$ be described on the surface of the immoveable sphere whose planes shall be at right angles to the plane of the circle $\overset{'''''}{p} \overset{'''''}{p} \overset{'''''}{p} \&c.$ the points $\overset{'}{q}, \overset{''}{q}, \overset{'''}{q}, \&c.$

therein, each 90° distant from $P, P, P, \&c.$ respectively, will be in a circle ($qqq \&c.$) parallel to the said circle $PPP \&c.$ Now as a regulation to the direction in which the force F shall urge the momentary pole, let that direction be always a tangent to the great circle so passing through that pole and the correspondent point $q, q, \text{ or } q, \&c.$ whilst the arcs $qqq, qqq, \&c.$ are to the arcs $PPP, PPP, \&c.$ respectively in the constant ratio of u to v .

The direction in which the force F acts being so regulated, it is obvious that the radius of the circle $PPP \&c.$ being denoted by b , the radius of the circle $qqq \&c.$ will be $= \sqrt{r^2 - b^2}$, the distance of these parallel circles being 90° . Therefore their peripheries being as the velocities (v and u) with which they are described, their radii (b and $\sqrt{r^2 - b^2}$) will be in the ratio of the said velocities; that is $v : u :: b : \sqrt{r^2 - b^2}$; whence, $\frac{u}{v}$ being $= \frac{\sqrt{r^2 - b^2}}{b}$, b , the radius of the circle $PPP \&c.$ is found $= \frac{rv}{\sqrt{u^2 + v^2}}$
 $= \frac{r}{\sqrt{1 + \frac{u^2}{v^2}}}$; and $\sqrt{r^2 - b^2}$ the radius of the circle $qqq \&c.$
 $\&c. = \frac{ru}{\sqrt{u^2 + v^2}} = \frac{r}{\sqrt{1 + \frac{v^2}{u^2}}}$, v being $= \frac{rF}{e}$, the velocity where-

with

with the momentary pole $\overset{''}{p}, \overset{'''}{p}$, &c. changes its place.

Consequently, if $\overset{\cdot}{P}R$ be an arc in the said immoveable concave sphere whose sine is $\frac{rv}{\sqrt{u^2+v^2}} = \frac{r}{\sqrt{1+\frac{e^2 u^2}{r^2 F^2}}}$, the great

circles $\overset{\cdot}{q}P, \overset{''}{q}P, \overset{'''}{q}P$, &c. will intersect each other at the point R .

7. Moreover, the force F being invariable and acting as expressed in the preceding article, the primitive pole $\overset{\cdot}{p}$ and the momentary poles $\overset{''}{p}, \overset{'''}{p}$, &c. will all be found in a circle $\overset{\cdot}{p}\overset{''}{p}\overset{'''}{p}$ &c. described upon the surface of the revolving sphere, as observed in that article; which circle, during the action of the force of F , will (as is also observed in the said article) always touch and roll along the immoveable circle ($\overset{\cdot}{P}P\overset{''}{P}$ &c.) whose radius we have just now found $= \frac{rv}{\sqrt{u^2+v^2}} = \frac{r}{\sqrt{1+\frac{e^2 u^2}{r^2 F^2}}}$; the point of contact being always the momentary pole.

Let the sine of the arc $\overset{\cdot}{P}Q$ of the great circle $\overset{\cdot}{R}\overset{\cdot}{p}\overset{\cdot}{Q}\overset{\cdot}{T}$ in the revolving sphere be equal to k , the radius of the said circle $\overset{\cdot}{p}\overset{''}{p}\overset{'''}{p}$ &c. then will the point Q and its opposite point (o) in the surface of the said sphere, during the action of the force F , describe circles in the sur-

rounding immoveable concave sphere parallel to ($\overset{'}{P}\overset{''}{P}\overset{'''}{P}$ &c.) the circle described by the momentary pole $\overset{''}{p}, \overset{'''}{p}$, &c. in the same concave sphere. And such point Q and its opposite point (o) being continually urged by the force F in directions at right angles to the tangents to the arcs they describe, their velocity will continue the same as before the action of the said force commenced; which velocity, and the radius of the said circle $\overset{''}{p}\overset{'''}{p}$ &c. will be determined by the following computation.

That radius being denoted by k , we have $r : k :: e : \frac{ek}{r}$, the velocity of the point Q before the action of the force F commenced; and $b : v :: \kappa : \frac{\kappa v}{b}$, the velocity of the same point (Q) during the action of that force, κ being put for the sine of the arc QR; therefore the velocity of Q continuing the same during the action of F as before, we have $\frac{ek}{r} = \frac{\kappa v}{b}$. But κ is the sine of the sum of the arcs

RP, PQ, whose sines are b and k respectively; therefore

$\frac{b\sqrt{r^2-k^2}}{r} + \frac{k\sqrt{r^2-b^2}}{r}$ will be $= \kappa$; and by substitution we get

$$\frac{ek}{r} = \frac{v\sqrt{r^2-k^2}}{r} + \frac{k v \sqrt{r^2-b^2}}{r} = \frac{v\sqrt{r^2-k^2}}{r} + \frac{ku}{r}, \frac{\sqrt{r^2-b^2}}{b} \text{ being } = \frac{u}{v}$$

by the preceding article. Hence we find $k =$

$$rv$$

$\frac{rv}{\sqrt{e-u^2+v^2}}$; and it follows, that $\frac{ev}{\sqrt{e-u^2+v^2}} (= \frac{ek}{r})$ will be equal to the velocity of the point Q, and likewise of its opposite point (o) in the surface of the sphere. It also follows, that κ , the radius of each of the circles described by those points, during the action of the force F will be equal to $\frac{rev}{\sqrt{u^2+v^2} \times \sqrt{e-u^2+v^2}}$.

By what is done it appears, that during the action of the force F, the motion of the revolving sphere will be regulated by the circle $\overset{''''''}{ppp}$ &c. thereon (whose radius is $\frac{rv}{\sqrt{e-u^2+v^2}} = \sqrt{1 + \frac{e^2 e-u^2}{r^2 \kappa^2}})$ continually touching and rol-

ling along the immoveable circle $\overset{''''''}{PPP}$ &c. (whose radius is $\frac{rv}{\sqrt{u^2+v^2}} = \sqrt{1 + \frac{e^2 u^2}{r^2 \kappa^2}})$ so that the velocity of the

point of contract be $= v = \frac{rF}{e}$. Considering the point Q as always urged from the points $\overset{''''''}{P, P, P}$, &c. and consequently its opposite point (o) towards those points, it is necessary to observe, that according as u is less or greater than e , the arc PQ (whose sine is $\frac{rv}{\sqrt{e-u^2+v^2}})$ will be less or greater than 90° ; and the point (o) opposite to Q on the surface of the sphere will accordingly be at a greater or less distance than 90° from P.

If

If u be negative the arc PR whose sine is $\frac{rv}{\sqrt{u^2 + v^2}}$ will be greater than 90° .

8. The motion of the sphere according to the regulation in the preceding article is one motion compounded of the primitive motion of the sphere and the motion generated by the action of the force F . But conceiving $\left(\frac{ev}{\sqrt{c-u}^2 + v^2}\right)$ the velocity of the point Q to arise from an impulse given to it whilst the sphere revolved about an axis of which Q was an immoveable pole before such impulse, and about which the mid-circle corresponding to that primitive axis revolved with the angular velocity $\frac{e \cdot e \cdot D \cdot u}{\sqrt{c-u}^2 + v^2} (a)$; and considering that the force F , continually acting at right angles to the momentary direction of the point Q and to the plane of the said mid-circle, only serves to alter the position of the said primitive axis; we may, by the help of what is done above, explain the motion which the sphere will have, during the action of the force F , so as to retain in our ideas the two primitive motions (one about the axis QO , and the other about a diameter at right angles to that axis) as remaining distinct and unaltered.

(a) Denoting this by e and the velocity of Q by d , $\sqrt{e^2 + d^2}$ is $= e$, agreeable to art. I.

Fig.

Fig. 6. Let ED be a great circle on the revolving sphere, of which Q is a pole, and let a smaller circle DL parallel to (MQ) that which we have found will be described by the point Q, be drawn on the immoveable concave sphere so as to touch that great circle in the point (D) where the great circle QPR cuts it; the radius of

which lesser circle will be $(= \sqrt{r^2 - K^2} =) \frac{rv^2 \sin r u \cdot \overline{e-u}}{\sqrt{u^2 + v^2} \times \sqrt{e-u}^2 + v^2}$.

Then the revolving sphere, during the action of the force F, will so move, that the first mentioned great circle (ED) shall continually touch and roll along the said lesser circle DL, the velocity of the point of contact (along that

circle) being $= \frac{v^2 \sin u \cdot \overline{e-u}^{(b)}}{\sqrt{e-u}^2 + v^2}$, and the sphere at the same

time turning about the axis of which Q is a pole with the primitive angular velocity $\frac{\overline{e \cdot e \sin u}}{\sqrt{e-u}^2 + v^2}$.

Thus the primitive motion about the axis of which Q is a pole is preserved distinct, whilst that pole proceeds describing a circle, whose radius is $\frac{rv}{\sqrt{u^2 + v^2} \times \sqrt{e-u}^2 + v^2}$,

with the velocity $\frac{ev}{\sqrt{e-u}^2 + v^2}$ which we supposed given to it.

(b) This is to the velocity of the point Q as $\sqrt{r^2 - K^2}$ to K; that is, as the radii of the arcs described.

It is observable, that the last mentioned velocity will, according to this regulation of the motion, be to the primitive angular velocity about the axis of which Q is a pole, as v to $e-u$, or as v to $u-e$, according as u is less or greater than e ; that is, according as the arc PQ is less or greater than 90° .

9. From what has been said it follows, that denoting the two primitive angular velocities $\frac{e \cdot \sin u}{\sqrt{e-u^2+v^2}}$ and $\frac{ev}{\sqrt{e-u^2+v^2}}$ (specified in the preceding article) by c and d respectively, the radius (fig. 5.) of the circle ppp &c. (or sine of the arc $PQ=PQ$, &c.) will be $= \frac{dr}{e}$; the radius of the circle PPP &c. (or sine of the arc $PR=PR$, &c.) $= \frac{dr^2 F}{e \sqrt{d^2 e^2 \mp 2cd r F + r^2 F^2}}$: a great circle passing through the primitive poles o and Q , on the revolving sphere, will turn from the position ORQ with the velocity $\frac{rF}{d}$ measured at the mid-circle, or with the velocity $\frac{drF}{\sqrt{d^2 e^2 \mp 2cd r F + r^2 F^2}}$ measured at the fixed point R ; whilst those poles describe, with the velocity d , circles parallel to PPP &c. the radius (κ) of each of the circles (fig. 6.) so described being

being $= \frac{d^2 r}{\sqrt{d^2 e^2 \mp 2 e d r F + r^2 F^2}}$: the radius $(\sqrt{r^2 - k^2})$ of the circle DL will be $= \frac{r \times c d + r F}{\sqrt{d^2 e^2 \mp 2 e d r F + r^2 F^2}}$; and the velocity $\left(\frac{v^2 \propto e. \overline{e-u}}{\sqrt{e-u}^2 + v^2} \right)$ along the said circle DL $= c + \frac{r F}{d}$: the upper or lower of the double signs taking place according as $u (= e \mp \frac{c r F}{d e})$ is less or greater than e ; that is, according as the arc PQ (whose sine is $= \frac{d r}{e}$) is less or greater than 90° .

10. As an instance of the use of the preceding conclusions, I will now apply them in the solution of a very interesting problem, which I have not before seen solved.

Suppose a given spheroid, whilst revolving uniformly about its proper axis, with a given angular velocity, to be suddenly urged by some percussive force to turn, with some given angular velocity, about a diameter of its equator; it is proposed to explain the rotatory motion of the spheroid consequent to the impulse so received.

Fig. 7, 8. Let DOEQ be the spheroid, whose femi-axis $co = cq$ is $= b$, and equatorial radius $cd = ce = r$; and supposing it before the impulse to revolve about its proper axis oq with the angular velocity c , measured at the distance r from the axis, let the poles (o and q) be suddenly urged by some percussive force to turn about a

diameter of the equator of the spheroid, with the angular velocity d , likewise measured at the distance r from that diameter. Upon receiving such impulse, the spheroid will take a new axis of motion, which will be a momentary one; suppose such new axis to be $p c \pi^{(c)}$. Then the particles of the spheroid being urged (or having a tendency) to turn about that axis with the angular velocity $\sqrt{c^2 + d^2}$, (which we will denote by e) their joint centrifugal force will so urge the spheroid to turn about that diameter of the equator which shall be at right angles to the momentary axis $p c \pi$, that the accelerative force of the point D of the equator to turn it about the said diameter according to the order of the letters $D Q E$ will (as appears by what is proved in art. 1. and in the Appendix annexed hereto) be $= \frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$ or $\frac{cd}{r} \times \frac{b^2 - r^2}{r^2 + b^2}$ according as b is less or greater than r : and it follows from hence and what is proved in art. 3. and 4. that v , the angular velocity (at the distance r from c) with which the momentary pole p will change its place, will accordingly be $= \frac{cd}{s} \times \frac{r^2 - b^2}{r^2 + b^2}$ or $\frac{cd}{e} \times \frac{b^2 - r^2}{r^2 + b^2}$.

(c) To find the position of this axis see art. 1. by which the sine of the angle $o c p$ (to the radius r) is found $= \frac{dr}{e}$.

More

Moreover, referring to our observation in art. 8. let $u-e$ be to $\frac{cd}{e} \times \frac{r^2-b^2}{r^2+b^2}$ (the value of v) as c to d , u being greater than e ; or let $e-u$ be to $\frac{cd}{e} \times \frac{b^2-r^2}{r^2+b^2}$ as c to d , u being less than e : whence, in both cases, we shall have the same expression $\left(\frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}\right)$ for the value of $u-e$; and consequently u , in both cases, will be $= e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$.

Conceive now a spherical surface without matter, having the same center and radius as the equator DE, to be carried about with the revolving spheroid; and suppose a sphere, whose radius is r , to revolve about an axis $p\pi$ with the angular velocity e , and, whilst it is so revolving, let an accelerative force (F) equal to $\frac{cd}{r} \times \frac{r^2-b^2}{r^2+b^2}$ or $\frac{cd}{r} \times \frac{b^2-r^2}{r^2+b^2}$, according as b is less or greater than r , urge the pole p , and the successive momentary poles as they become such, to turn about a diameter of the contemporary mid-circle in the manner expressed in art. 6. u being to v as $e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$ to $\frac{cd}{e} \times \frac{r^2-b^2}{r^2+b^2}$ or as $e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$ to $\frac{cd}{e} \times \frac{b^2-r^2}{r^2+b^2}$, according as b is less, or greater than r . Then will the motion of the surface of this sphere be exactly the same as the motion of the said spherical surface carried about with the revolving spheroid

spheroid after receiving the impulse of the percussive force. Therefore, having reference to our conclusions in the preceding articles, we, by substitution, readily obtain solution to our problem.

By substituting properly $\frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$ or $\frac{cd}{r} \times \frac{b^2 - r^2}{r^2 + b^2}$ for F , we find,

$$\frac{d^2 r}{\sqrt{d^2 e^2 \mp 2cd r F + r^2 F^2}} = \frac{dr}{c} \times \frac{r^2 + b^2}{\sqrt{4r^4 + r^2 + b^2}^2 \times \frac{d^2}{c^2}},$$

$$\frac{\frac{r \times cd + rF}{\sqrt{d^2 e^2 \mp 2cd r F + r^2 F^2}}}{\frac{r \times cd + rF}{\sqrt{d^2 e^2 \mp 2cd r F + r^2 F^2}}} = \frac{2r^3}{\sqrt{4r^4 + r^2 + b^2}^2 \times \frac{d^2}{c^2}},$$

$$\text{and } c + \frac{rF}{d} = \frac{2r^2 c}{r^2 + b^2}.$$

Which equations, respect being had to the conclusions in art. 8. and 9. indicate that, whether b be less or greater than r , if an immoveable circle DL , whose radius is =

$$\frac{2r^3}{\sqrt{4r^4 + r^2 + b^2}^2 \times \frac{d^2}{c^2}}, \text{ be conceived to be described in a plane}$$

inclined to the plane of the equator of the spheroid (before the impulse) in an angle whose sine (to the radius r)

$$\text{is } = \frac{dr}{c} \times \frac{r^2 + b^2}{\sqrt{4r^4 + r^2 + b^2}^2 \times \frac{d^2}{c^2}}, \text{ so that the said circle touch}$$

the said equator in the point D in the section $o p D Q E$; the spheroid after the impulse will so revolve, that its
equator

equator will always touch and roll along the said immoveable circle (DL), the velocity of the point of contact (along that circle) being $= \frac{2r^2c}{r^2+b^2}$, whilst the spheroid turns about its proper axis (OQ) with the primitive angular velocity c , and the poles o and Q (by the said rolling of the equator) describe circles (whose radii are each $= \frac{bd}{c} \times \sqrt{\frac{r^2+b^2}{4r^4+r^2+b^2} \times \frac{d^2}{c^2}}$) parallel to the said circle.

DL, with the angular velocity d (or their proper velocity $\frac{bd}{r}$) which we supposed given to them by the impulse^(d). Thus the motion of the spheroid consequent to the impulse appears to be remarkably regular.

And in the very same manner may be explained the motion of a cylinder, whose primitive motion about its proper axis may be disturbed by some percussive force in like manner as we supposed the spheroid disturbed; only (instead of the former substitution for F) substituting for the accelerative force arising from the centrifugal force of the particles of the revolving cylinder its proper value $\frac{cd}{r} \times \frac{3r^2-4b^2}{3r^2+4b^2}$ (computed in our Appendix) and afterwards proceeding as we have done with regard to the spheroid,

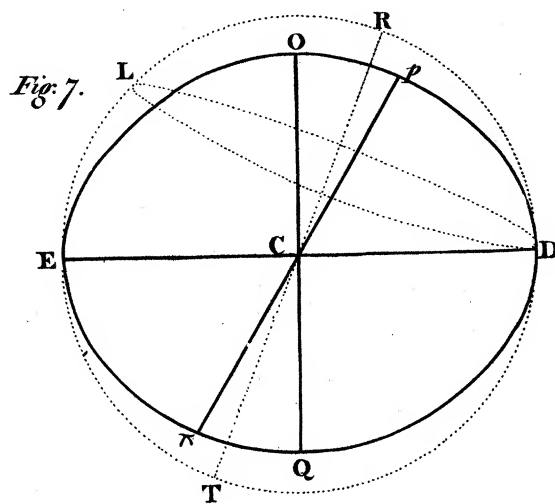
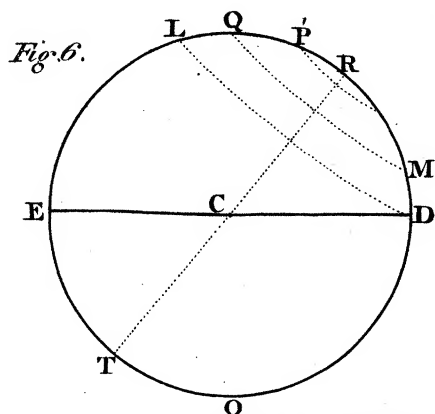
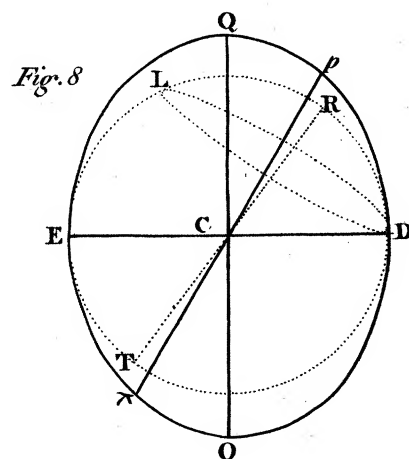
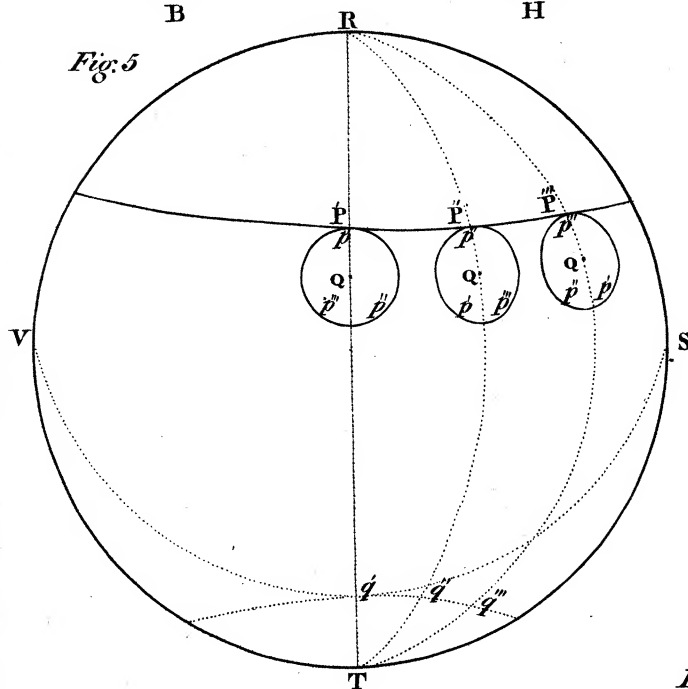
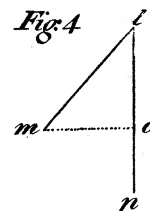
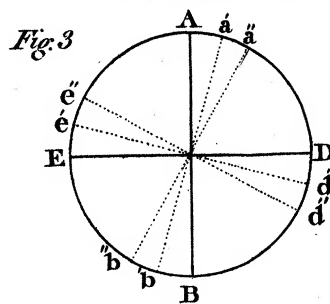
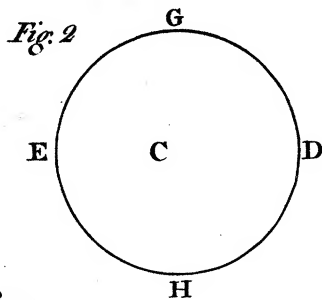
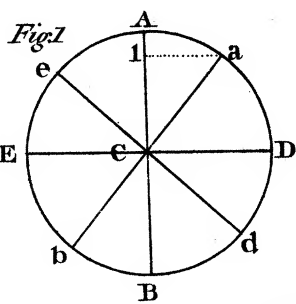
(d) Other ways of solving the problem are also suggested by the preceding articles.

b denoting half the length of the cylinder, and r the radius of any section at right angles to its proper axis.

Seeing that $\left(\frac{cd}{r} \times \frac{3r^2 - 4b^2}{3r^2 + 4b^2}\right)$ the expression for the said accelerative force respecting the cylinder vanishes when b is $= \frac{3^{\frac{1}{2}}r}{2}$, it is manifest that the cylinder in that case will (with respect to its own particles) undisturbedly revolve about any axis whatever passing through its center of gravity, as will a sphere. Which remarkable property of that particular cylinder I believe has not before been taken notice of.

There are, I am aware, bodies of other forms having the like property.

The preceding articles lead us to consider the motion of the earth's axis in a light, I presume, more clear and satisfactory than any in which it has before been considered; but I must, for want of leisure, defer making the application till some future opportunity; only observing here, that by what is done above it appears, that from the action of the Sun and Moon on the earth its axis has a diurnal motion, which I have no where seen explained. Which motion is not much unlike that of the axis of the revolving spheroid just now considered, when $(2b)$ this last mentioned axis is many times longer than $(2r)$



the equatorial diameter of the said spheroid, and $\frac{d}{c}$ very small.

A P P E N D I X.

Shewing how the joint centrifugal force of the particles of a spheroid or cylinder, having a rotatory motion about any momentary axis, is computed.

1. FIG. 9. Let p be a particle of matter firmly connected with the plane $DOEFQG$, in which the line OCQ is situated; and pq being a perpendicular from p to the said plane, let the distance pq be denoted by u ; also, the line ql being at right angles to OCQ , let the distance pl be denoted by b . Then, the said plane with the particle p being made to revolve about OCQ as an axis, with the angular velocity e measured at the distance a from the said axis, the velocity of p will be $= \frac{be}{a}$, and its centrifugal force from l will (by a well-known theorem) be $= \frac{be^2}{a^2}$ to make it a^2 the expression being $\frac{be^2}{a^2} \times p$. Whence, by resolving that force into two others, one in the direction qp , and the other in a direction parallel to lq , it appears that the force urging p from the plane $DOEFQG$ will be $= \frac{ue^2}{a^2} \times p$, let the distance lq be what it will.

2. The particle p being connected with the plane $DOEFQG$ as mentioned in the preceding article, and the distance cl being denoted by v ; if p be urged directly from the said plane by a force $fu \times p$, the efficacy of that force to turn the said plane about the line HCI , therein drawn at right angles to ocq , will (by the property of the lever) be equivalent to the force $\frac{fuv \times p}{g}$ acting on the said line ocq at right angles to the said plane at the distance g from the point c .

Moreover it is obvious, that, *cæteris paribus*, the efficacy will be the same let the distance of q from l be what it will.

Fig. 10. Let q coincide with l ; and let ck be a line in the plane c/p continued (which plane will be at right angles to the plane $DOEFQG$); also, pk being at right angles to ck , let those lines pk and ck be denoted by w and x respectively. Then the sine and cosine of the angle kco to the radius r , being respectively denoted by m and n , the force $\frac{fuv \times p}{g}$ will be $= \frac{f \times p}{g} \times m n \times w^2 - x^2 + m^2 - n^2 \times wx$.

Consequently, if each particle of any solid body, through which a line HCI and a plain $DOEIFQGH$ may be conceived to pass, be urged from that plane by a force expressed by $fu \times p$ as above; the force which, acting on the line ocq at the distance g from c , would be equivalent to the efficacy

efficacy of all the forces acting on the several particles of that body to turn the same about the line HC will be obtained by computing the sum of all the forces

$\frac{f \times p}{g} \times mn \times w^2 - x^2 + m^2 - n^2 \times wx$ acting on the said body.

The computation of such equivalent force will in most cases be abridged by observing that, if pk be continued to p'' so that kp'' be $= kp$, the efficacy of the force on the particle p'' , to turn the body about the line HC in opposition to the force on the particle p , will be represented by

the equivalent force $\frac{f \times p''}{g} \times mn \times x^2 - w^2 + m^2 - n^2 \times wx$ acting on the line OCQ at the distance g from c ; and that therefore the efficacy of the two forces on p and p'' , to turn the body about HC , will be represented by the equivalent force $\frac{2f \times p}{g} \times mn \times w^2 - x^2$ acting on the line OCQ , at right angles to the plane $DOEIFQGH$, at the distance g from c .

3. Fig. 11, 12. If the body be a cylinder, a spheroid, or the like, and its proper axis be situated in the line ck , the ordinates corresponding the *abscissæ* kp , kp'' , in the circular section bi whose center is k , will each be parallel to that diameter passing through c , about which the body will be urged to turn; and each of those ordinates will be $= \sqrt{y^2 - w^2}$, y being the radius of such section.

Therefore, writing $2\sqrt{y^2-w^2}$ instead of p , it follows that

$\frac{4Af}{g} \times mn \times \sqrt{\frac{y^4}{4} - x^2 y^2}$, the whole fluent of $\frac{4fx\sqrt{y^2-w^2}}{g} \times$

$mn \times \sqrt{w^2 - x^2} \times w$, generated $w (=kp=k\dot{p})$ from 0 becomes equal to the radius y (both x and y being considered as invariable) will express the value of the force which, acting on the line ocq at the distance g from c , would be equivalent to the force of all the particles in the said section, whose thickness is denoted by the indefinitely small quantity $\dot{x}$; the distance ck being denoted by x , and A being put for (.78539) the area of a quadrant of a circle whose radius is 1.

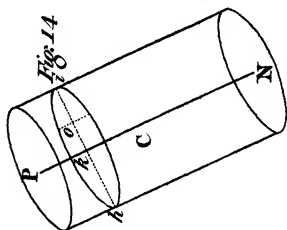
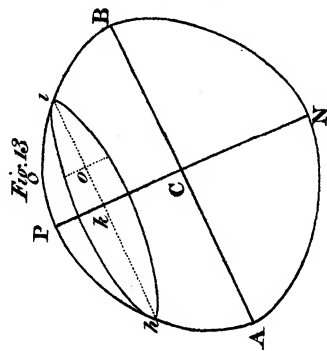
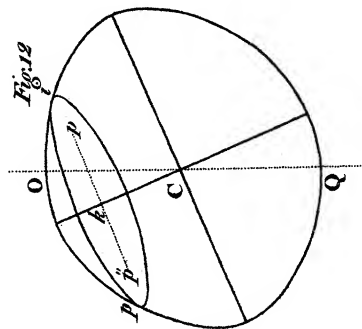
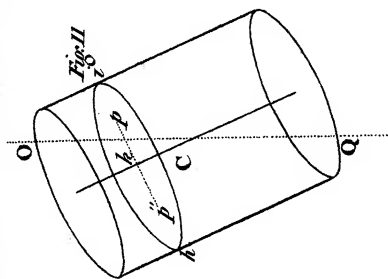
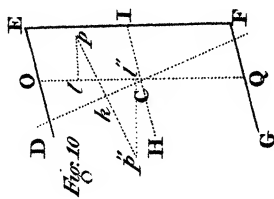
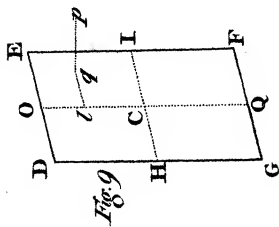
4. Fig. 11. In the cylinder whose length is $2b$ and diameter $2r$; y being $r, \frac{y^4}{4} - x^2 y^2$ will be $r^2 \times \frac{r^2}{4} - x^2$: consequently, the fluent of $\frac{r^2}{4} - x^2 \times \dot{x}$, generated whilst x from 0 becomes $=b$, being $\frac{br^2}{4} - \frac{b^3}{3}$, we have $\frac{8Af r^2}{g} \times mn \times \frac{br^2}{4} - \frac{b^3}{3} = \frac{f mn}{12g} \times \sqrt{3r^2 - 4b^2} \times M$ for the force which, acting as above at the distance g from (c) the center of gravity of the cylinder, would be equivalent to the efficacy of the forces acting as above on all the particles of the cylinder to turn it about a diameter passing through c , M being the mass or content of the cylinder.

5. Fig. 12. In the spheroid whose proper axis is $2b$ and equatorial diameter $2r$, y^2 being $= \frac{r^2}{b^2} \times \overline{b^2 - x^2}$, $\frac{y^4}{4} - x^2 y^2$ will be $= r^2 \times \frac{r^2}{4} - \frac{r^2 x^2}{2b^2} + \frac{r^4 x^2}{4b^4} - x^2 + \frac{x^4}{b^2}$: consequently, the fluent of $\frac{r^2 x}{4} - \frac{r^2 x^2}{2b^2} + \frac{r^4 x^2}{4b^4} - x^2 x + \frac{x^4 x}{b^2}$, generated whilst x from 0 becomes $= b$, being $\frac{r^2 b}{4} - \frac{r^2 b}{6} + \frac{r^2 b}{20} - \frac{b^3}{3} + \frac{b^3}{5} = \frac{2}{15} \times \overline{r^2 b - b^3}$, we have $\frac{16 \Delta f r^2}{15g} \times mn \times \overline{r^2 b - b^3} = \frac{f m n}{5g} \times \overline{r^2 - b^2} \times s$ for the force which, acting at the distance g from c the center of the spheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the spheroid to turn it about a diameter of its equator, s being the mass or content of the spheroid.

These equivalent forces are distinguished by the name of motive forces; the correspondent accelerative forces are computed in the following articles.

6. Fig. 13. The body being a spheroid whose center is c , and whose proper axis PN is $= 2b$ and equatorial diameter $AB = 2r$; let F be the accelerative force of a particle at the distance g from the axis about which the body is urged to turn, which axis is supposed to be a diameter of its equator. Denote ck by x ; ki by y ; and let the abscissa ko and its correspondent ordinate (parallel to the last mentioned axis) in the circle whose radius is ki be denoted

denoted by s and z respectively. Then, considering the body as urged to turn about that diameter of its equator which is at right angles to AB , the accelerative force of every particle in the said ordinate will be $= \frac{\sqrt{s^2 + x^2}}{g} \times F$, and the motive force of all the particles in the same ordinate will be $= \frac{\sqrt{s^2 + x^2}}{g} \times F z s = \frac{\sqrt{s^2 + x^2}}{g} \times F s \sqrt{y^2 - s^2}$; to which (by the property of the lever) a motive force $= \frac{s^2 + x^2}{g^2} \times F s \sqrt{y^2 - s^2}$ acting at the distance g from the center at right angles to a ray therefrom would be equivalent. Therefore, considering x and y as invariable, and s only as variable, $\frac{4Fx}{g^2} \times$ the whole fluent of $s \sqrt{y^2 - s^2} \times \sqrt{s^2 + x^2}$ will denote a force which, acting at the distance g from c , would be equivalent to the motive force of all the particles in the section bi whose radius is ki and thickness x . Which fluent is $= Ay^2 \times x^2 + \frac{y^3}{4} = \frac{Ay^2}{b^2} \times \overline{b^2 - x^2} \times x^2 + \frac{r}{4b^2} \times \overline{b^2 - x^2}$. Consequently $\frac{8Ay^2F}{b^2g^2} \times$ the whole fluent of $x \times \overline{b^2 - x^2} \times \overline{x^2 + \frac{r^2}{4b^2} \times \overline{b^2 - x^2}}$ will denote a motive force which, acting at the distance g from c at right angles to a ray therefrom, would be equivalent to the whole motive force urging the spheroid to turn as above mentioned. Such equivalent force



force will therefore be $= \frac{16 A r^2 F}{15 g^2} \times \overline{r^4 b + r^2 b^3} = \frac{F}{5 g^2} \times \overline{r^2 + b^2} \times s$;

and this being put $= \frac{f m n}{5 g} \times \overline{r^2 - b^2} \times s$ (the value of the

same force found in art. 5.) we find $F = f g m n \times \frac{r^2 - b^2}{r^2 + b^2}$;

which will be $= \frac{g m n e^2}{a^2} \times \frac{r^2 - b^2}{r^2 + b^2}$, if f be $= \frac{e^2}{a^2}$, its value computed in art. 1.

Or F will be denoted by $\frac{e d}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$; if r be to e as m to d , and as n to c ; and a and g be each $= r$.

7. Fig. 14. The body being a cylinder whose center of gravity is in c , and whose proper axis PN is $2b$ and diameter $2r$; the accelerative force (F) at the distance g from c , will in like manner be found $= \frac{g m n e^2}{a^2} \times \frac{3r^2 - 4b^2}{3r^2 + 4b^2}$; the cylinder being considered as urged to turn about a diameter passing through c .

If $r : e :: m : d :: n : c$, and a and g be each $= r$, F will be $= \frac{e d}{r} \times \frac{3r^2 - 4b^2}{3r^2 + 4b^2}$.

